12.7

Another Method

Completing the Square

LEARNING GOALS

In this lesson, you will:

- Determine the roots of a quadratic equation by completing the square.
- Complete the square geometrically and algebraically.

KEY TERMS

- completing the square

Can you construct a square that has the exact same area as a given circle using only a compass and a straightedge?

Well, no. And this was proven to be impossible in 1882, making pi a "transcendental" irrational number.

Unfortunately, it seems that no one in Indiana got the message at the time. In 1897, the Indiana state legislature, via amateur mathematician Edwin Goodwin, tried to pass a law declaring that there was a solution to this famous problem—known as completing the square.

PROBLEM 1 Where Are the Zeros?

1. Factor each quadratic function and determine the zeros, if possible.

   a. $f(x) = x^2 + 5x + 4$

   b. $f(x) = x^2 - 4x + 2$

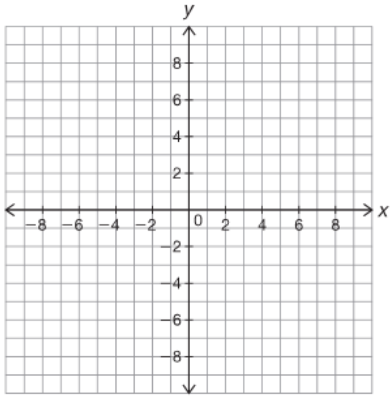
   c. $f(x) = x^2 + 5x + 2$

   d. $f(x) = x^2 - 4x - 5$

2. Were you able to determine the zeros of each function in Question 1 by factoring? Explain your reasoning.

3. If you cannot factor a quadratic function, does that mean it does not have zeros?

a. Graph the quadratic function from Question 1, part (b) on your calculator. Sketch the graph on the coordinate plane.



b. Does this function have zeros? Explain your reasoning.

The quadratic function you graphed has zeros but cannot be factored, so we must find another method for calculating its zeros. You can use your understanding of the relationship among the coefficients of a perfect square trinomial to construct a procedure to solve any quadratic equation.

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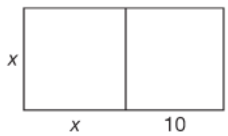
PROBLEM 2 Seeing the Square

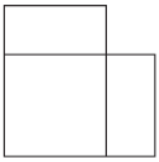
Previously, you factored trinomials of the form $a^2 + 2ab + b^2$ as the perfect square $(a + b)^2$. This knowledge can help you when constructing a procedure for solving any quadratic equation.

1. The expression $x^2 + 10x$ can be represented geometrically as shown. Write the area of each piece in the center of the piece.



2. This figure can now be modified into the shape of a square by splitting the second rectangle in half and rearranging the pieces.

   a. Label the side length of each piece and write the area of each piece in the center of the piece.



   b. Do the two figures represent the same expression? Explain your reasoning.

   c. Complete the figure so it is a square. Label the area of the piece you added.

   d. Add this area to your original expression. What is the new expression?

   e. Factor this expression.

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The process you worked through above is a method known as *completing the square*. **Completing the square** is a process for writing a quadratic expression in vertex form which then allows you to solve for the zeros.

3. Use a geometric figure to complete the square for each expression. Then factor the resulting trinomial.

   a. $x^2 + 8x$

   b. $x^2 + 5x$

4. Analyze your work in Question 3.

   a. Explain how to complete the square on an expression $x^2 + bx$ where b is an integer.

   b. Describe how the coefficient of the middle term, b, is related to the constant term, c in each trinomial you wrote in Question 3.

5. Use the descriptions you provided in Question 4 to determine the unknown value of b or c that would make each expression a perfect square trinomial. Then write the expression as a binomial squared.

   a. $x^2 - 8x +$ ______ = ______

   b. $x^2 + 5x +$ ______ = ______

   c. $x^2 -$ ______ $+ 100 =$ ______

   d. $x^2 +$ ______ $+ 144 =$ ______

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So how does completing the square help when trying to determine the roots of a quadratic equation that cannot be factored? Let's take a look.

Determine the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$. You can make this into a perfect square trinomial.	$x^2 - 4x + 2 - 2 = 0 - 2$ $x^2 - 4x = -2$
Determine the constant term that would complete the square. Add this term to both sides of the equation.	$x^2 - 4x + \_\_\_\_ = -2 + \_\_\_\_$ $x^2 - 4x + 4 = -2 + 4$ $x^2 - 4x + 4 = 2$
Factor the left side of the equation.	$(x - 2)^2 = 2$
Determine the square root of each side of the equation.	$\sqrt{(x - 2)^2} = \pm\sqrt{2}$ $x - 2 = \pm\sqrt{2}$
Set the factor of the perfect square trinomial equal to each of the square roots of the constant. Solve for x.	$x - 2 = \pm\sqrt{2}$ $x - 2 = \sqrt{2}$ or $x - 2 = -\sqrt{2}$ $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$ $x \approx 3.414$ or $x \approx 0.5858$ The roots are approximately 3.41 and 0.59.

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6. Check the solutions in the worked example by substituting each into the original equation.

7. Do your solutions match the zeros you sketched on your graph in Problem 1, Question 3? Explain how you determined your answer.

8. Determine the roots of each equation by completing the square.

a. $x^2 - 6x + 4 = 0$

b. $x^2 - 12x + 6 = 0$

You can identify the axis of symmetry and the vertex of any quadratic function written in standard form by completing the square.

$$y = ax^2 + bx + c$$

Step 1: $y - c = ax^2 + bx$

Step 2: $y - c = a\left(x^2 + \frac{b}{a}x\right)$

Step 3: $y - c + a\left(\frac{b}{2a}\right)^2 = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right)$

Step 4: $y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Step 5: $y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2$

Step 6: $y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

Notice that the a-value was factored out before completing the square!

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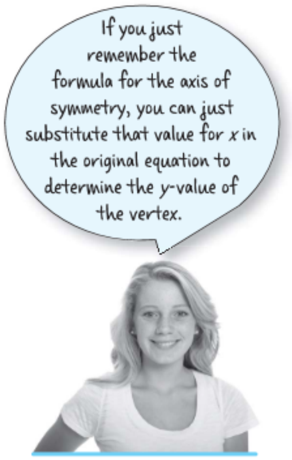
9. Explain why $a\left(\frac{b}{2a}\right)^2$ was added to the left side of the equation in Step 3.

10. Given a quadratic function in the form $y = ax^2 + bx + c$,

a. identify the axis of symmetry.

b. identify the location of the vertex.



11. Rewrite each quadratic equation in vertex form. Then identify the axis of symmetry and the location of the vertex in each.

a. $y = x^2 + 8x - 9$

b. $y = 3x^2 + 2x - 1$

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12. Determine the roots and the location of the vertex of $y = x^2 + 20x + 36 = 0$.

Be prepared to share your solutions and methods.